A Non-Zero Sum Game
Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>B Cooperates</th>
<th>B Defects</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Cooperates</td>
<td>-1 , -1</td>
<td>-9 , 0</td>
</tr>
<tr>
<td>A Defects</td>
<td>0 , -9</td>
<td>-6 , -6</td>
</tr>
</tbody>
</table>

Non-Zero-Sum means there’s at least one outcome in which (A’s PAYOFF + B’s PAYOFF) ≠ 0
Normal Form Representation of a Non-Zero-Sum Game with $n$ players

Is a set of $n$ strategy spaces $S_1, S_2 \ldots S_n$
where $S_i =$ The set of strategies available to player $i$

And $n$ payoff functions
$u_1, u_2 \ldots u_n$
where $u_i : S_1 \times S_2 \times \ldots S_n \to \mathbb{R}$
is a function that takes a combination of strategies (one for each player) and returns the payoff for player $i$

<table>
<thead>
<tr>
<th></th>
<th>PLAYER B (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLAYER A (1)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>-1 , -1</td>
</tr>
<tr>
<td>D</td>
<td>0 , -9</td>
</tr>
</tbody>
</table>

$n = 2$
$S_1 = \{C, D\}$
$S_2 = \{C, D\}$

what would you do if you were Player A ??
Strict Domination

If one of a player’s strategies is never the right thing to do, no matter what the opponents do, then it is **Strictly Dominated**

If player A plays “C”, what should I do?

If player A plays “D”, what oh what should I do?

Fundamental assumption of game theory:

**Get Rid of the Strictly Dominated strategies. They Won’t Happen.**

In some cases (e.g. prisoner’s dilemma) this means, if players are “rational” we can predict the outcome of the game.
“Understanding” a Game

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“Understanding” a Game

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<td>-9 , 0</td>
</tr>
<tr>
<td>D</td>
<td>0 , -9</td>
<td>-6 , -6</td>
</tr>
</tbody>
</table>

In some cases (e.g. prisoner’s dilemma) this means, if players are “rational” we can predict the outcome of the game.

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Non-Zero-Sum Game Theory: Slide 9
So is strict domination the best tool for predicting what will transpire in a game?

<table>
<thead>
<tr>
<th>Player A</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>3,1</td>
<td>4,1</td>
<td>5,9</td>
<td>2,6</td>
</tr>
<tr>
<td>II</td>
<td>5,3</td>
<td>5,8</td>
<td>9,7</td>
<td>9,3</td>
</tr>
<tr>
<td>III</td>
<td>2,3</td>
<td>8,4</td>
<td>6,2</td>
<td>6,3</td>
</tr>
<tr>
<td>IV</td>
<td>3,8</td>
<td>3,1</td>
<td>2,3</td>
<td>4,5</td>
</tr>
</tbody>
</table>

Strict Domination doesn’t capture the whole picture

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0,4</td>
<td>4,0</td>
</tr>
<tr>
<td>II</td>
<td>4,0</td>
<td>0,4</td>
</tr>
<tr>
<td>III</td>
<td>3,5</td>
<td>3,5</td>
</tr>
</tbody>
</table>

What strict domination eliminations can we do?

What would you predict the players of this game would do?
Nash Equilibria

\[ S_i^* \in S_i, S_2^* \in S_2, \cdots S_n^* \in S_n \]

are a NASH EQUILIBRIUM iff

\[ \forall i \quad S_i^* = \arg \max_{S_i} u_i(S_1^*, S_2^*, \cdots S_{i-1}^*, S_i, S_{i+1}^* \cdots S_n^*) \]

\[ \begin{array}{ccc}
I_a & I_b & II_b \\
\hline
I_a & 0 & 4 & 4 & 0 & 5 & 3 \\
II_a & 4 & 0 & 0 & 4 & 5 & 3 \\
III_a & 3 & 5 & 3 & 5 & 6 & 6 \\
\end{array} \]

(III_a,III_b) is a N.E. because

\[ u_i(III_a,III_b) = \max \left[ u_i(I_a,III_b), u_i(II_a,III_b), u_i(III_a,III_b) \right] \]

AND \[ u_2(III_a,III_b) = \max \left[ u_2(III_a,I_b), u_2(III_a,II_b), u_2(III_a,III_b) \right] \]
• If \((S_1^*, S_2^*)\) is an N.E. then player 1 won’t want to change their play given player 2 is doing \(S_2^*\)
• If \((S_1^*, S_2^*)\) is an N.E. then player 2 won’t want to change their play given player 1 is doing \(S_1^*\)

Find the NEs:

\[
\begin{array}{ccc|ccc}
-1 & -1 & -9 & 0 & 4 & 0 \\
0 & -9 & -6 & -6 & 4 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc|ccc}
0 & 4 & 5 & 3 & 4 & 0 \\
3 & 5 & 3 & 6 & 6 \\
\end{array}
\]

• Is there always at least one NE?
• Can there be more than one NE?

Example with no NEs among the pure strategies:

\[
\begin{array}{ccc}
S_1 & S_2 \\
S_1 & \\
S_2 & \\
\end{array}
\]
Example with no NEs among the pure strategies:

<table>
<thead>
<tr>
<th></th>
<th>S₁</th>
<th>S₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>S₁</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>S₂</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

2-player mixed strategy Nash Equilibrium

The pair of mixed strategies \((M_A, M_B)\) are a Nash Equilibrium iff

- \(M_A\) is player A’s best mixed strategy response to \(M_B\)
- \(M_B\) is player B’s best mixed strategy response to \(M_A\)
**Fundamental Theorems**

- In the $n$-player pure strategy game $G=\{S_1, S_2, \cdots, S_n; u_1, u_2, \cdots, u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(S_1^*, S_2^* \cdots S_n^*)$ then these strategies are the unique NE of the game.

- Any NE will survive iterated elimination of strictly dominated strategies.

- [Nash, 1950] If $n$ is finite and $S_i$ is finite $\forall i$, then there exists at least one NE (possibly involving mixed strategies).

---

**The “What to do in Pittsburgh on a Saturday afternoon” game**

Pat enjoys football  
Chris enjoys hockey  
Pat and Chris are friends: they enjoy spending time together

\[
\begin{array}{c|cc}
\text{Chris} & H & F \\
\hline
H & 1 & 0 \\
F & 0 & 2 \\
\end{array}
\]

- Two Nash Equilibria.
- How useful is Game Theory in this case??
- Why this example is troubling…
INTERMISSION

(Why) are Nash Equilibria useful for A.I. researchers?

Will our algorithms ever need to play…

Prisoner’s Dilemma?
Saturday Afternoon?

Nash Equilibria Being Useful

THE TRAGEDY OF THE Commons

• You graze goats on the commons to eventually fatten up and sell
• The more goats you graze the less well fed they are
• And so the less money you get when you sell them
Commons Facts

How many goats would one rational farmer choose to graze?

What would the farmer earn?

What about a group of $n$ individual farmers?

Answering this…

…is good practice for answering this

$n$ farmers

$i$'th farmer has an infinite space of strategies

$$g_i \in [0, 36]$$

An outcome of

$$(g_1, g_2, g_3, \ldots, g_n)$$

will pay how much to the $i$'th farmer?
\( n \) farmers

\( i \)'th farmer has an infinite space of strategies

\[ g_i \in [0, 36] \]

An outcome of

\[ (g_1, g_2, g_3, \ldots, g_n) \]

will pay how much to the \( i \)'th farmer?

\[ g_i \times \sqrt{36 - \sum_{j=1}^{n} g_j} \]

Let's **Assume a pure Nash Equilibrium** exists.

Call it \( (g_1^*, g_2^*, \ldots, g_n^*) \)

What can we say about \( g_1^* \)?

\[ g_i^* = \arg \max_{s_i} \text{Payoff to farmer } i, \text{ assuming } \]

\[ \text{the other players play } \]

\[ (g_1^*, g_2^*, \ldots, g_{i-1}^*, g_{i+1}^*, \ldots, g_n^*) \]

For Notational Convenience,

write \( G_{-i}^* = \sum_{j \neq i} g_j^* \)

THEN

\[ g_i^* = \arg \max_{s_i} \left[ \right. \]

\[ \text{What?} \]
Let's Assume a pure Nash Equilibrium exists.

Call it
\[
\left( g_1^*, g_2^*, \ldots, g_n^* \right)
\]

What can we say about \( g_i^* \)?

Payoff to farmer i, assuming the other players play
\[
\left( g_1^*, g_2^*, \ldots, g_{i-1}^*, g_{i+1}^*, \ldots, g_n^* \right)
\]

For Notational Convenience,
write \( G_{-i}^* = \sum_{j \neq i} g_j^* \)

THEN
\[
g_i^* = \arg \max_{s_i} \left[ g_i \sqrt{36 - g_i - G_{-i}^*} \right]
\]

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Non-Zero-Sum Game Theory: Slide 27
We have \( n \) linear equations in \( n \) unknowns

\[
\begin{align*}
g_1^* &= 24 - \frac{2}{3}(g_2^* + g_3^* + \cdots g_n^*) \\
g_2^* &= 24 - \frac{2}{3}(g_1^* + g_3^* + \cdots g_n^*) \\
g_3^* &= 24 - \frac{2}{3}(g_1^* + g_2^* + g_4^* + \cdots g_n^*) \\
&\vdots \\
g_n^* &= 24 - \frac{2}{3}(g_1^* + \cdots + g_{n-1}^*)
\end{align*}
\]

Clearly all the \( g_i^* \)'s are the same (Proof by “it’s bloody obvious”)

Write \( g^* = g_1^* = \cdots = g_n^* \)

Solution to \( g^* = 24 - \frac{2}{3}(n-1)g^* \) is:

\[
g^* = \frac{72}{2n+1}
\]

Consequences

At the Nash Equilibrium a rational farmer grazes \( \frac{72}{2n+1} \) goats.

How many goats in general will be grazed? Trivial algebra gives: \( \frac{36}{2n+1} \) goats total being grazed

[as \( n \to \text{infinity} \), \( \#\text{goats} \to 36 \)]

How much profit per farmer?

\[
\frac{432}{(2n+1)^{3/2}}
\]

1.26¢ if 24 farmers

How much if the farmers could all cooperate?

\[
\frac{24\sqrt{n}}{n} \cdot \frac{83}{n} = \frac{3.46}{n}
\]

3.46¢ if 24 farmers
The Tragedy

The farmers act “rationally” and earn 1.26 cents each. But if they’d all just got together and decided “one goat each” they’d have got 3.46 cents each.

Is there a bug in Game Theory?
   in the Farmers?
   in Nash?

Would you recommend the farmers hire a police force?

Recipe for Nash-Equilibrium-Based Analysis of Such Games

• Assume you’ve been given a problem where the $i$’th player chooses a real number $x_i$
• Guess the existence of a Nash equilibrium $(x_1^*, x_2^* \cdots x_n^*)$
• Note that, $\forall i,$
  
  $x_i^* = \text{arg max}_x \begin{cases} \text{Payoff to player } i \text{ if player } i \\ \text{plays } x_i^* \text{ for } j \neq i \\ \text{plays } x^*_{j} \text{ if } j \neq i \end{cases}$

• Hack the algebra, often using “at $x_i^*$ we have $x_i = ?$ Payoff + 0 “
**INTERMISSION**

Does the Tragedy of the Commons matter to us when we’re building intelligent machines?

Maybe repeated play means we can learn to cooperate??

---

**Repeated Games with Implausible Threats**

Takeo and Randy are stuck in an elevator
Takeo has a $1000 bill
Randy has a stick of dynamite
Randy says “Give me $1000 or I’ll blow us both up.”

<table>
<thead>
<tr>
<th></th>
<th>Takeo: -1000</th>
<th>Takeo: -10^7</th>
<th>Takeo: 0</th>
<th>Takeo: -10^7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Randy:</td>
<td>1000</td>
<td>10^7</td>
<td>0</td>
<td>10^7</td>
</tr>
</tbody>
</table>

What should Takeo do??????
Using the formalism of Repeated Games With Implausible Threats, Takeo should **Not** give the money to Randy

Takeo Assumes Randy is Rational

At this node, Randy will choose the left branch

Repeated Games

Suppose you have a game which you are going to play a finite number of times.

What should you do?

---

### 2-Step Prisoner’s Dilemma

**GAME 1**

<table>
<thead>
<tr>
<th></th>
<th>Player B C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player A C</td>
<td>-1 , -1</td>
<td>-9 , 0</td>
</tr>
<tr>
<td>D</td>
<td>0 , -9</td>
<td>-6 , -6</td>
</tr>
</tbody>
</table>

**GAME 2**

(Played with knowledge of outcome of GAME 1)

<table>
<thead>
<tr>
<th></th>
<th>Player B C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player A C</td>
<td>-1 , -1</td>
<td>-9 , 0</td>
</tr>
<tr>
<td>D</td>
<td>0 , -9</td>
<td>-6 , -6</td>
</tr>
</tbody>
</table>

Player A has four pure strategies

- C then C
- C then D
- D then C
- D then D

**Is Idea 1 correct?**

Ditto for B
Important Theoretical Result:

Assuming Implausible Threats, if the game G has a unique N.E. \((s_1^*, \ldots, s_n^*)\) then the new game of repeating G \(T\) times, and adding payouts, has a unique N.E. of repeatedly choosing the original N.E. \((s_1^*, \ldots, s_n^*)\) in every game.

If you’re about to play prisoner’s dilemma 20 times, you should defect 20 times.

DRAT 😔

Intermission

Game theory has been cute so far.
But depressing.
Now let’s make it really work for us.
We’re going to get more real.
The notation’s growing teeth.
Bayesian Games

You are Player A in the following game. What should you do?

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>3 ?</td>
<td>-2 ?</td>
</tr>
<tr>
<td>S2</td>
<td>0 ?</td>
<td>6 ?</td>
</tr>
</tbody>
</table>

Question: When does this situation arise?

Hockey lovers get 2 units for watching hockey, and 1 unit for watching football.

Football lovers get 2 units for watching football, and 1 unit for watching hockey.

Pat’s a hockey lover.

Pat thinks Chris is probably a hockey lover also, but Pat is not sure.

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pat</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

With 2/3 chance

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pat</td>
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<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
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1/3 chance

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Non-Zero-Sum Game Theory: Slide 39

Non-Zero-Sum Game Theory: Slide 40
In a Bayesian Game each player is given a type. All players know their own types but only a prob. dist. for their opponent’s types.

An $n$-player Bayesian Game has:

- A set of action spaces $A_1 \cdots A_n$
- A set of type spaces $T_1 \cdots T_n$
- A set of beliefs $P_1 \cdots P_n$
- A set of payoff functions $u_1 \cdots u_n$

$P_i(t_i|t_i)$ is the prob dist of the types for the other players, given player $i$ has type $t_i$.

$u_i(a_1, a_2 \cdots a_n, t_i)$ is the payout to player $i$ if player $j$ chooses action $a_j$ (with $a_j \in A_j$) (forall $j=1,2,\cdots n$) and if player $i$ has type $t_i \in T_i$.

---

**Bayesian Games: Who Knows What?**

We assume that all players enter knowing the full information about the $A_i$’s, $T_i$’s, $P_i$’s and $u_i$’s

The $i$’th player knows $t_n$, but not $t_1 t_2 t_3 \cdots t_{i-1} t_{i+1} \cdots t_n$

All players know that all other players know the above.

And they know that they know that they know, *ad infinitum*.

**Definition**: A strategy $S_i(t_i)$ in a Bayesian Game is a mapping from $T_i$ to $A_i$, a specification of what action would be taken for each type.
Example

\[ A_1 = \{H,F\} \quad A_2 = \{H,F\} \]
\[ T_1 = \{\text{H-love}, \text{Flove}\} \quad T_2 = \{\text{Hlove}, \text{Flove}\} \]
\[
\begin{align*}
P_1 (t_2 = \text{Hlove} | t_1 = \text{Hlove}) &= 2/3 \\
P_1 (t_2 = \text{Flove} | t_1 = \text{Hlove}) &= 1/3 \\
P_1 (t_2 = \text{Hlove} | t_1 = \text{Flove}) &= 2/3 \\
P_1 (t_2 = \text{Flove} | t_1 = \text{Flove}) &= 1/3 \\
P_2 (t_1 = \text{Hlove} | t_2 = \text{Hlove}) &= 1 \\
P_2 (t_1 = \text{Flove} | t_2 = \text{Hlove}) &= 0 \\
P_2 (t_1 = \text{Hlove} | t_2 = \text{Flove}) &= 1 \\
P_2 (t_1 = \text{Flove} | t_2 = \text{Flove}) &= 0 \\
\end{align*}
\]

\[ u_1 (H,H,Hlove) = 2 \quad u_2 (H,H,Hlove) = 2 \]
\[ u_1 (H,H,Flove) = 1 \quad u_2 (H,H,Flove) = 1 \]
\[ u_1 (H,F,Hlove) = 0 \quad u_2 (H,F,Hlove) = 0 \]
\[ u_1 (F,H,Hlove) = 0 \quad u_2 (F,H,Hlove) = 0 \]
\[ u_1 (F,H,Flove) = 0 \quad u_2 (F,H,Flove) = 0 \]
\[ u_1 (F,F,Hlove) = 1 \quad u_2 (F,F,Hlove) = 1 \]
\[ u_1 (F,F,Flove) = 2 \quad u_2 (F,F,Flove) = 2 \]

\[ \text{Bayesian Nash Equilibrium} \]

The set of strategies \((s_1^*, s_2^* \cdots s_n^*)\) are a

Pure Strategy Bayesian Nash Equilibrium

iff for each player \(i\), and for each possible type of \(i\): \(t_i \in T_i\)

\[ s_i^*(t_i) = \arg \max \sum_{a_i \in A_i} u_i (s_i^*(t_i), \ldots s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots s_n^*(t_n)) \times P_i (t_i | t_i) \]

i.e. no player, in any of their types, wants to change their strategy
NEGOTIATION: A Bayesian Game

Two players: S, (seller) and B, (buyer)

$T_s = [0,1]$ the seller’s type is a real number between 0 and 1 specifying the value (in dollars) to them of the object they are selling

$T_b = [0,1]$ the buyer’s type is also a real number. The value to the buyer.

Assume that at the start

$V_s \in T_s$ is chosen uniformly at random

$V_b \in T_b$ is chosen uniformly at random

The “Double Auction” Negotiation

S writes down a price for the item $(g_s)$
B simultaneously writes down a price $(g_b)$
Prices are revealed

If $g_s = g_b$ no trade occurs, both players have payoff 0

If $g_s = g_b$ then buyer pays the midpoint price $(g_s + g_b)/2$ and receives the item

Payoff to S: $1/2(g_s + g_b) - V_s$
Payoff to B: $V_b - 1/2(g_s + g_b)$
Negotiation in Bayesian Game Notation

\[ T_s = [0,1] \text{ write } V_s \in T_s \]
\[ T_b = [0,1] \text{ write } V_b \in T_b \]
\[ P_s(V_b|V_s) = P_s(V_b) = \text{uniform distribution on } [0,1] \]
\[ P_b(V_s|V_b) = P_b(V_s) = \text{uniform distribution on } [0,1] \]
\[ A_s = [0,1] \text{ write } g_s \in A_s \]
\[ A_b = [0,1] \text{ write } g_b \in A_b \]
\[ u_s(P_s,P_b,V_s) = \text{What?} \]
\[ u_b(P_s,P_b,V_b) = \text{What?} \]

Double Negotiation: When does trade occur?

...when
\[ g_b^*(V_b) = 1/12 + 2/3 V_b > 1/4 + 2/3 V_s = g_s^*(V_s) \]
i.e. when \[ V_b > V_s + 1/4 \]

\[ \begin{array}{c}
? \\
V_s \\
\frac{1}{4} \\
\frac{1}{2} \\
1
\end{array} \quad \begin{array}{c}
V_b \\
\frac{1}{4} \\
\frac{1}{2} \\
\frac{3}{4} \\
1
\end{array} \]

\[ \text{Prob(Trade Happens)} = 1/2 \times (3/4)^2 = 9/32 \]
Value of Trade

\[ E[V_s | \text{Trade Occurs}] = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4} \]
\[ E[V_b | \text{Trade Occurs}] = \frac{1}{4} + \frac{2}{3} \times \frac{3}{4} = \frac{3}{4} \]

If trade occurs, expected trade price is

\[ \frac{1}{2} g_s*(V_s) + g_b*(V_b) = \frac{1}{2}(\frac{1}{12} + \frac{2}{3}V_b + \frac{1}{4} + \frac{2}{3}V_s) = \frac{1}{6} + \frac{1}{3}V_b + \frac{1}{3}V_s \]

Value of Trade continued...

\[ E[\text{profit to S | trade occurred } ] = \]
\[ E[\frac{1}{6} + \frac{1}{3}V_b + \frac{1}{3}V_s - V_s | \text{trade occurred } ] = \]
\[ \frac{1}{6} + \frac{1}{3}E[V_b | \text{trade } ] - \frac{2}{3}E[V_s | \text{trade } ] = \]
\[ \frac{1}{6} + \frac{1}{3} \times \frac{3}{4} - \frac{2}{3} \times \frac{1}{4} = \frac{1}{4} \]

Similar Algebra Shows: \( E[\text{profit to B | trade occurred } ] = \frac{1}{4} \) also

<table>
<thead>
<tr>
<th>Using This Game</th>
<th>If Both Were “Honest”</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E[ B’s \text{ profit } ] = ) 1/4x9/32=0.07</td>
<td>( E[ B \text{ profit } ]=1/12=0.083 )</td>
</tr>
<tr>
<td>( E[ S’s \text{ profit } ] = ) 0.07</td>
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</tbody>
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This Game seems **Inefficient**. What can be done???
Double Auction: Final Comments

• There are other Nash Equilibrium strategies.
• But the one we saw is provably most efficient.
• In general, even for arbitrary prob. dists. of $V_s$ and $V_b$, no efficient NE’s can exist.
• And no other games for this kind of trading can exist and be efficient.

Double Auction Discussion

What if seller used “giant eagle” tactics?
Seller states “I’ll sell it to you for price $p$ : take it or leave it”

Exercise:
• How should* seller choose price (taking into account $V_s$ of course) ?
• And how should* buyer choose whether to buy ? *(at a B.N.E.)
• When could/should double auction technology be used?
• (How) can “$V_s, V_b$ drawn randomly from [0,1]” be relaxed ?
First Price Sealed Bid

Seller wants to sell an object that has no value to seller… anything seller is paid is pure profit.

There are \( n \) available buyers

Assumptions:

- Assume buyer \( i \) has a value for the object distributed uniformly randomly in \([0,1]\)
- Assume \( V_i \)'s all independent
- Buyer \( i \) does not know \( V_j \) for \( i \neq j \)
- Buyer \( i \) knows all \( V_j \)'s randomly generated from \([0,1]\)
First Price Sealed Bid Rules

Each buyer writes down their bid.

Call buyer $i$’s bid $g_i$

Buyer who wrote highest bid must buy object from seller at price=bid

**Question:** Why is “bid = $V_i$” a stupid strategy??

Auction Analysis: Back to Bayesian Nash Equils

We’ll assume that all players other than $i$ do a linear strategy:

$$g_j^*(V_j) = m_j V_j \text{ for } j \neq i$$

Then what should $i$ do?
\[ g^*_i(v_i) = \arg \max_g E[\text{Profit if play } g] \]

\[ = \arg \max_g E[\text{Profit if } i \text{ wins} \cdot \text{Prob } g \text{ is winning bid}] \]

\[ = \arg \max_g \left( \frac{\text{what?}}{\text{what?}} \right) \times \left( \frac{\text{what?}}{\text{what?}} \right) \]

\[ = g \text{ such that } (n-1)(v_i - g)g^{n-2} - g^{n-1} = 0 \]

\[ \Rightarrow g^*_i(v_i) = \left(1 - \frac{1}{n}\right)v_i \]
Thus we’ve an N.E. because if all other players use a linear strategy then it’s in $i$’s interest to do so too. Above holds $\forall i$

First-Price Sealed Auction

At BNE all players use

$$g_i^*(v_i) = (1-1/n)V_i$$

Note: [Fact of probability]

Expected value of the largest of $n$ numbers drawn independently from $[0,1]$ is $\frac{n}{n+1}$

Expected profit to seller = what?
First-Price Sealed Auction

At BNE all players use
\[ g^*_i(V_i) = (1-1/n)V_i \]

Note: [Fact of probability]
Expected value of the largest of \( n \) numbers drawn independently from \([0,1]\) is \( \frac{n}{n+1} \)

Expected profit to seller =

Expected highest bid = what?

\[ \left( 1 - \frac{1}{n} \right) \left( \frac{n}{n+1} \right) = 1 - \frac{2}{n+1} \]

Exercise: compute expected profit to player \( i \). Show it is \( O(1/n) \).
Second-Price Sealed Bid

A different game:
Each buyer writes their bid
Buyer with highest bid must buy the object
But the price they pay is the second highest bid
• What is player $i$’s best strategy
• Why?
• What is seller’s expected profit?

Auction Comments

• Second-price auction is preferred by cognoscenti
  ➢ No more efficient
  ➢ But general purpose
  ➢ And computationally better
  ➢ And less variance (better risk management)
• Auction design is interesting
  ➢ So far mostly for economics
  ➢ But soon for e-commerce etc.?
• Important but not covered here
  ➢ Expertise
  ➢ Collusion
  ➢ Combinatoric Auctions
  ➢ What if all cooperative ????
What You Should Know

Strict dominance
Nash Equilibria
Continuous games like Tragedy of the Commons
Rough, vague, appreciation of threats
Bayesian Game formulation
Double Auction
1st/2nd Price auctions

What You Shouldn’t Know

• How many goats your lecturer has on his property
• What strategy Mephistopheles uses in his negotiations
• What strategy this University employs when setting tuition
• How to square a circle using only compass and straight edge
• How many of your friends and colleagues are active Santa informants, and how critical they’ve been of your obvious failings